"IS THIS A FIVE MINUTE ARGUMENT OR THE FULL HALF HOUR?"[2] ENRICHING CLASSROOMS WITH A CULTURE OF REASONING

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Reasoning is one of the proficiency strands highlighted in the *Australian Curriculum: Mathematics*. This is because it is the heart of mathematical work. However, because the content strands receive the bulk of the explicit description in the document, it can be difficult to know how to ensure that reasoning plays a key role in the classroom. This paper presents strategies to enhance the nature and extent of reasoning activity taking place in classrooms, and discusses how to build a culture of justification, use questioning techniques to encourage reasoning, and develop tasks that require higher order thinking.

Introduction

An anecdote from the distant past

My mind is taken back to 1978. I am in Year 9, in a maths classroom in a standard 1950s-built Tasmanian high school. It is the advanced maths class, where we get extra maths as an elective, on top of regular maths classes. There are about 28 students in the class, only four of them girls (typical for the time), and we are being taught by Mr Smith who is the senior master for maths. We have been doing Pythagoras' theorem, and, in this time where calculators are only just being introduced into schools, all of the answers to the problems are beautiful whole numbers. Thus we meet the famous (3,4,5) triangle, and the (6,8,10) triangle (and can see the connection between the two, although we explore that no further); and we also meet some of the other Pythagorean triples like (5,12,13), (7,24,25), and (9,40,41).

At this point, someone notices something about the numbers in our triples: the first is odd, and the final pair are consecutive. Mr Smith encourages us to test this and we find (11,60,61), giving our newfangled calculators a workout in the process. Someone else notices that in the sequence 12, 24, 40, 60—formed by the second numbers in our sequence of triples—the difference between each pair of consecutive numbers goes up by 4 as 12, 16, 20. This leads us to conjecture that (13,84,85) is another of these special triples. We reach again for our calculators (a thrill to use but it still feels like cheating), and the new example is confirmed.

2 From *Monty Python's Flying Circus*, episode 29.

We try to articulate what we have discovered, and thus "9CHAM's Theorem[3]" is born. It asserts that for every odd number there is a Pythagorean triple—not that we know this name at this time—where the remaining two numbers are consecutive. Mr Smith is as excited about this as we are; it is new to him as well. There is a small amount of frustration, however, because although we are confident about the result and can see the patterns, we cannot quite articulate it mathematically. I think, too, that a few of us recognise that, in fact, we cannot be absolutely sure that this result always holds, partly because we have not been able to nail it down properly, and partly because we think it needs a proof like the ones we had been doing in geometry with a QED thingy at the end.

Commentary from the present

Well, my memory is hazy, and I confess I cannot be sure that the events above occurred in exactly the way I have described, but most of it is roughly accurate, and I *do* remember being excited during the discovery of 9CHAM's Theorem. There was a buzz in the classroom as we spotted patterns, and conjectured, and tested, and tried to explain what was happening. I do not know if *everyone* in the class felt it, or followed the arguments or all of the discussion, but it seemed as if they did. Mr Smith allowed us to conduct the investigation—indeed, he investigated with us (no, I do not think he just pretended for our sakes), and he encouraged the conjecturing, testing, and articulating that led us to the final result. Although this was the most striking example, there was generally always a positive culture of discussion and reasoning in that classroom.

Two years later my Year 11 teacher showed me the proof of 9CHAM's Theorem. Yes, it really is true (it is also relatively well-known, though not to my Year 9 class and Mr Smith). As it happens, there is even more to it. It was nice to have the result properly articulated and proven, but I do recall a vague disappointment that I had not been able to prove it for myself. (To be fair, I do not think I had really tried to prove the result for myself in the intervening years.) Partly this was because the Year 11 teacher's proof was actually of an even more general result, and I could not fully link it to the specific cases that we had encountered in 9CHAM's Theorem, which made my Year 11 teacher's proof not exactly the one I thought I was looking for. In hindsight, though, I am wondering how much I *could* have done, with the right guidance. I am not sure I would have been able to determine the necessary algebraic formulation of the patterns, but I *do* think I could have completed, for myself, the algebraic manipulation that proves the result … and I also think I could have been scaffolded to find the algebraic formulation of the 9CHAM's Theorem special cases. These experiences raise important questions about what reasoning is possible in classrooms and how to get it to happen.

Reasoning and the curriculum

It is now 2013. I am no longer a high school student; my job now is to prepare prospective teachers to teach high school mathematics. What should they be teaching? What should be going on in *their* Year 9 classrooms?

3 CHAM = Clarence High Advanced Mathematics, and, yes, I know now that at the time it was only a conjecture … but you should tell that to everyone who ever talked about Fermat's Last "Theorem" before 1994.

The last 35 years have seen the demise of geometric proof from the curriculum, which has been bemoaned by those who see it as epitomizing a central tenet of the discipline of mathematics: namely deductive reasoning. On the other hand, for others the loss is unlamented, since many students struggled with it, it appears to have little value to real world applications, and its removal allowed space in the curriculum to emphasise other aspects of mathematical activity and content. Whatever your view of the place of geometric proof in the curriculum, however, I do not believe that the reduced emphasis on it was *intended* to imply a reduction in reasoning in the curriculum. Nevertheless, this may be what happened. Stacey (2003), in examining the results of the TIMSS 1999 video study, discussed evidence for a "shallow teaching syndrome" in Australian Year 8 mathematics classrooms. She pointed out the excessive use of repetition in the sequences of problems assigned to students, the use of problems of low complexity, and the absence of mathematical reasoning in the activities of the classroom, and suggested that these may play a role in the differences in outcomes from international testing between Australian students and their international counterparts.

Reasoning is certainly specified in curriculum documents as being an essential component of mathematics education. Like various curricula before it (e.g., Victorian Curriculum and Assessment Authority, 2008, with its "Working Mathematically" strand), the recently released *Australian Curriculum: Mathematics* (Australian Curriculum, Assessment and Reporting Authority, 2013) places emphasis on reasoning. It is one of four proficiency strands, together with fluency, understanding, and problem solving. In the curriculum document, these four strands are treated differently from the content strands number and algebra, measurement and geometry, and statistics and probability. In some respects this is not surprising, since content and proficiency are quite different from each other as aspects of mathematical knowledge. However, if teachers are looking to the curriculum document as a guide for what they are to teach, then there is potential for them to underestimate the importance of reasoning. Although I am not claiming that volume of text in a curriculum document is indicative of the weight that is intended to be given to a curriculum strand, it is rather telling that the vast majority of the *Australian Curriculum: Mathematics* is taken up with the content strands. In part this is because the content strands have tended to be encyclopaedic, spelling out all the topics to be covered. In contrast, the potential pervasiveness of reasoning within mathematics means that the curriculum indicates the scope of reasoning only by giving a few examples of where it might arise, rather than trying to specify all the places where it could be incorporated. This pervasiveness—and the limited collection of supplied examples—puts the onus on the teacher to identify places where reasoning can be highlighted. Furthermore, the fact that some aspects of mathematics can be (and sometimes are) "taught" in classrooms in the absence or limited presence of reasoning means that it is might be useful to look at strategies that can build a culture of reasoning in the classroom.

Strategies for fostering reasoning

My suggestions in what follows have been inspired by a number of sources, including the work of Watson and Mason on questions and prompts for the mathematics classrooms (1998) and constructing examples (2005); the work of Stein, Grover, and Henningsen on task selection and implementation (1996; and also Henningsen & Stein, 1997); the list of habits of mind posited by Cuoco, Goldenberg, and Mark (1996); and Swan's work on strategies for improving learning (2005); together with other long-forgotten sources over the years, including teachers and mentors. The suggestions are intended to provide strategies, approaches and ideas for how to use tasks and questions to foster a stronger culture of reasoning in the classroom.

Tasks

Optimising routine tasks

Routine 'exercises' are an important part of mathematical learning, not least because they build fluency and understanding. You can maximise their effectiveness for reasoning by discussing why the steps follow in sequence, and how starting off with the conditions in the problem statement leads to the conclusion or answer in a logical way based on mathematical properties.

Selecting tasks that foster reasoning

Look for tasks that require multiple steps to solve, allow multiple entry points, provide opportunities to make links to other results, can be extended (see further comments later), and require or allow the opportunity for conjecturing. Do not underestimate your students' capacity to engage in "hard" problems, in a supportive environment.

Maximise the reasoning afforded by a task

Sometimes it can be difficult to identify and develop all the reasoning opportunities inherent in a particular task. A classic example is the use of pattern-spotting activities (e.g., how many matches are there in the tenth diagram in the sequence |_|, |_|_|, |_|_|_|, |_|_|_|_|, ...?). Many students will be able to identify the pattern, but more work and reasoning is required to describe the pattern using a mathematical formulation, and to justify the validity of the formulation. This principle means that it is important for the teacher to have tried the task, and to have thought in advance about what learning opportunities it offers and how to bring these out in the classroom.

Adapting tasks

Look for ways to open up a task to make it amenable to greater opportunities for reasoning. Routine exercises often can be turned into richer tasks, with a little thought. Try 'reversing' the task, by taking the answer and posing new, related tasks with that answer (e.g., What other numbers only have 3 factors? Can you find other equations of lines that pass through (2, 3)? What other parallelograms could you construct if (1, 2) and (3, 7) are vertices on one side?). Try asking "what if?" questions (e.g., If that angle changed from 30° to 60° what would happen to the remaining angles in the triangle? What would happen to the volume if the dimensions of a prism were doubled?)

Reasoning in real-world tasks

Steen (1999) points out that formal deductive reasoning is needed only rarely for real world problems; but of course this is not the only type of reasoning that can and should be fostered in classrooms. Working with real world problems—with their ambiguity and the need for modelling and approximation—provides opportunities for reasoning while devising models that ensure a good match between the model and the real-world situation, examining the impact of variations in the model's assumptions, and checking the plausibility of numerical solutions.

Questions

There are many kinds of teacher and student questions that can foster reasoning. The distinction between a task and a question is sometimes unclear, and so there is some overlap among the categories of questions suggested in what follows and with some of the principles suggested earlier for developing and using tasks.

- Questions that ask for justification (e.g., Why can you perform that step? Can you explain why the mean changes by so much when you remove that outlier?)
- Questions that allow reflection on or development of the general principles that are evident in a specific problem. Asking for "another example" is a powerful way of doing this (see Watson & Mason, 2005) (e.g., Can you give me another example of a line parallel to $y = 3x$? Can you find an example of a number which is larger than its square? And another? And another?)
- Questions that encourage students to conjecture (e.g., Can you describe the pattern? What would happen if the coefficient of x^2 changed?)
- Questions that encourage students to test and question assumptions (e.g., Can you find an example that *doesn't* work? What if it *wasn't* a right-angled triangle?)
- Questions that encourage students to explore and experiment (e.g., How rare is it to get 5 heads (or tails) in a row when tossing a coin ten times? Do all quadrilaterals tessellate?). Comments from the discussion on pattern-spotting are relevant here: such experiments are usually only part of the story, and generally they should be a precursor to a more rigorous and reasoned analysis.

Classroom culture

It can take time to develop the culture of the classroom so that reasoning is encouraged and valued. Teachers will need to model reasoning and to ask questions requiring it. The classroom needs to be a supportive environment in which students can make conjectures and put forward their reasoning without risking their self-worth. At the same time, it is important for students to know that the ideas are testable and that questioning, validating, discussing, justifying, and disproving are all essential parts of the reasoning process.

Teachers can also encourage students to look for generalisations and to encourage a sense of wonder. This can be habituated by encouraging students to reflect on the tasks they have completed and ask themselves, "What happens if...?" or "Is there a sensible 'next question' I could ask about this situation?" A teacher's own enthusiasm for reasoning processes can also foster an environment in which reasoning is viewed as engaging and stimulating work, if not actually enjoyed and valued.

Other issues

The role of content knowledge and pedagogical content knowledge

Although these strategies may help to foster a culture of reasoning in the classroom, key roles are played by content knowledge and pedagogical content knowledge. A teacher needs mathematical knowledge to determine the potential of a task and the possible directions it might lead. Moreover, the choice of task alone is not enough. Stacey (2003) in reviewing the work of Stein and Henningsen and colleagues, points out that "although good tasks might seem to be the causal mechanism, the teacher influences the choice, timing, and detail of their implementation in classrooms. A lesson may be more or less successful in sustaining high level thinking, depending on the actions and pedagogical decisions of the teacher within the classroom" (p. 121). Sullivan, Clarke, and Clarke (2009) also discuss the challenges of turning a task into classroom activity that provides genuine learning opportunities. There are many mathematical and pedagogical decisions that a teacher needs to make in advance *and* in the hurly-burly of actual teaching in order to maximize reasoning opportunities. One particular challenge for the teacher is to interpret, validate, and counter students' reasoning as required; and also to engage in his or her own mathematical reasoning during the lesson, by constructing examples and counter-examples that may further develop learning.

Classroom management

As suggested earlier, building a culture of reasoning in the classroom requires that the teacher build an atmosphere of respect for students and for the idea of determining the usefulness or validity of ideas. This requires management of the discourse that take place, so that it is respectful of others, and opinions are backed by reasons or justification. Teachers should ensure that thinking time is allowed, and that students are given an opportunity to contribute regardless of the speed at which their idea was determined (it may be worth having a frank discussion with students about the merits and fairness of the strategies you use to select students to give their responses). Monitor students as they work and ask for their conjectures and reasons. This monitoring can also allow teachers to be strategic in choosing contributions from students. Use mini whiteboards as a scratchpad for working, or a place to record conjectures, or to vote for a conjecture (by having students "vote" they are forced to commit to one hypothesis over another; however, students must also learn that "popular opinion" is not actually an appropriate reasoning strategy).

An example

The following set of questions provides an example of a series of tasks and questions that will provide students with the opportunity to explore and reason, in this case with geometry. The sequence might be conducted with any grade level from 5 to 12. Naturally, you might expect different levels of rigour and/or formality in the reasoning from different groups of students, but, regardless of the level, the tasks allow scope for rich discussions, exploration, conjecturing, testing, and justifying.

The sequence was inspired from the single question "How many right angles can a pentagon have?" from Watson and Mason (1998).

Right angles in polygons

- How many right angles can a quadrilateral have? [Explore this. Have you covered all possibilities?]
- Can you have a quadrilateral with exactly two right angles, in which the angles are adjacent to each other? Why?
- Can you have a quadrilateral with exactly two right angles, in which the angles are opposite each other? Why?
- Can you have a quadrilateral with exactly three right angles? Why?

- Can you have a quadrilateral with one right angle and an obtuse internal angle?
- Can you have a quadrilateral with one right angle adjacent to an obtuse internal angle?
- How many right angles can a pentagon have? [Have you covered all possibilities? Did you only think of the regular pentagon at first?]
- Can you have a pentagon with exactly four right angles? Why?
- How many right angles can a hexagon have? [Have you covered all possibilities? Did you only think of convex hexagons at first?]
- Is it possible to get four right angles as the internal angles of a hexagon? Why?
- Is it possible to get more than four right angles as the internal angles of a hexagon? Why?
- Why does the hexagon *have* to be convex if you have four or more right angles?
- What's the next question that we could consider?

Conclusions

Building a culture of reasoning is not easy. It demands significant content and pedagogical content knowledge as well as classroom management skills. Yet there are rewards. Research suggests that student outcomes are better, which is already a favourable result; but the anecdotal evidence also suggests reason-filled classrooms have a vibrant atmosphere with engaged students in which the discipline of mathematics is not only explored and learned, but is actively experienced by students who are actually working mathematically.

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